same time speaking the corresponding number words. With some tribes the *use* of the fingers is the important thing, the accompanying vocal utterance being of secondary importance; e. g., some of the Eskimo tribes use the same *words* for 6, 7, 8, 9, 10 as for 1, 2, 3, 4, 5 but count them on the second hand. In others the language development has led to independent numerals which often preserve evidence of their digital origin. Examples are given in the next section. In widely separated tribes all over the continent actual finger counting has been observed and the rather remarkable fact noted that the order of counting is almost always uniform, commencing with the little finger of one hand and counting to the thumb, thence to the thumb of the other hand and to the little finger again. Usually the fingers are bent as the counting continues, but sometimes the hand is first clenched and the fingers then extended one at a time.[1] This general uniformity of order gives considerable aid in the linguistic discussion of the next paragraph.

(*c*) *Linguistic evidence.* The German philologist Grimm in speaking of the Old World languages says: "Alle Zahlwörter gehen aus von den Fingern der Hände."[2] Is this observation true for the New World? In the languages of civilized nations the numerals are so ground down from long usage that it is difficult to detect in their form their possible digital origin. While this is also the case in some Indian languages, in many there are striking similarities, while in others digital words and number words are almost or quite identical. Following the observed order of counting, the little fingers would be used for 1 and 10, the fourth fingers for 2 and 9, etc. Only a few typical examples of the many noted will be given with reference to each number.

The number ONE. Some of the names given the little finger are "the smallest," "the last of the hand," "little daughter of the hand." While we do not expect to find the word for *one* alway or even usually connected with that for little finger (since the concept of unity doubtless preceded formal counting) yet some instances are known; e. g., Massachusetts: *pasuk* from *piasuk*, "very small"; Montagnais: *inlare*, "end is bent"; Zuni: *topinte*, "taken to start with." Two is sometimes derived from finger; e. g., Montagnais: *nake*, "another bent in"; Dakota: *nonpa*, "to bend down"; Zuni: *kwillin*, "that (finger) put down with its like." But more often it is connected with the word for "hand," probably because there are two hands.[3] THREE. The third finger is often named "the middle"; e. g., Massachusetts: *nishwe*, from *nashaue*, "half way"; Zuni: *hain*, "equally dividing one." FIVE is counted on the thumb and we have Karankawan: *natsa behema*, from *natsa*, "one," *behema*, "finger." But more prominent is the idea that the hand is completed, variously expressed as "finished," "fingers finished," "all fingers," "all done," etc.; e. g., Ojibwa: *nanan*,

[1] Is this order the natural one? Many children have been observed to use their thumb first in counting. But Cantor has noted that the South African savages use the same order as the Indian order given above. See Cantor: *Vorlesungen über Geschichte der Mathematik*, Band I, Dritte Auflage, Leipzig, 1907, pp. 6–7.

[2] Grimm: *Geschichte der deutschen Sprache*, I, p. 167.

[3] Discussed more fully below under "Duplicative Principle."

"gone," "spent," and similarly in several Eastern languages. Hidatsa: *kichu*, from *ki*, "completely," *chu*, "turned down"; Ute: *munugi*, from *manoku*, "all." In most instances however it is connected with "hand," or "whole hand"; e. g., Kaniagmiut: *talgamen*, from *talega*, "hand"; Comanche: *mowaka*, from *mowa*, "hand"; Klamath: *tunep*, from *tu*, "away," *nep*, "hand," i. e., "hand-away."

From six to nine the numerals are expressed (*a*) from the names of the fingers used; (*b*) by "hand" + 1, 2, 3, 4; (*c*) by 1, 2, 3, 4, + "again" or "besides" (most frequent); or (*d*) by 1, 2, 3, 4 repeated without change. This last method is found only among the Eskimo and is probably always accompanied by the actual use of the second hand.

SIX. The Point Barrow six well illustrates the evolution of a number word. We are given the three forms

atautyimin-akbinigin-tudlimut, literally,	"once-on next- (and) five,"
atautyimin-akbinigin	"once-on next,"
akbinigin	"on next."

Other examples of six are Tano: *manli*, from *man*, "hand," *li*, "piece," i. e., "hand and piece of next"; Klamath: *nadshk-shapta*, "one I have bent over"; Takelma: *maimis*, "finger one in." SEVEN falls on the index finger or "pointer," e. g., Zuni: *tserucek* from *tserwerc*, "to point"; Greenland: *arfinek-mardluk*, "on the other hand-two"; Omaha: *penompa*, from *pe*, "finger," *nompa*, "two." For EIGHT, Hudson's Bay: *kittukleemot*, "middle finger"; Omaha: *pethatbathi*, "finger-three"; Klamath: *ndan-kshapta*, "three I have bent over." For NINE the subtractive principle comes into use and we have the additional forms "one left," "only one," etc.[1] We also have the forms, Greenland: *mikkelerak*, "fourth finger"; Zuni: *tenalikya*, "all but one held up with rest." TEN is counted on the little finger, e. g., Hudson's Bay: *eerkitkoka*, "little finger." But more prominent is the fact "two hands completed," "man finished," or "man." Thus Zuni: *astemthla*, "all of the fingers"; Wintun: *pampa-sempta* from *pampu-ta*, "two," *sem*, "hand"; Konkau: *machoko*, from *mar*, "hand," *choko*, "double."

Above ten, various combinations of the first ten numerals occur in which of course these digital names reappear. A few other examples of interest will be given. ELEVEN. Unalit: *atkhakhtok*, "it goes down" (referring to change from hands to feet). THIRTEEN. Greenland: *arkanerpingasut*, "on the first foot, three," etc. SIXTEEN. Unalit: *gukhtok*, "it goes over" (to toes of other foot). NINETEEN. Maidu: *tsoi-ni-maiduk*, from *maidu*, "man," *tsoi*, "four"—"four with man," i. e., after 15, 4 on toward 20 (man); and similarly for 16, 17, 18. TWENTY. In decimal-system languages twenty is usually but not always "two tens." In the vigesimal it is quite commonly "man," "Indian," "all hands and feet"; e. g., Navaho: *natin*, from *tine*, "man"; Greenland: *inuk-navdlugo*, "man come to an end," or *inup-avatai-navdlugit*, "man's outer members completed"; Kaniagmiut: *swinuk*, from *suk*, *innuk*, "man"; Wintun: *ketet-wintun*, from *ketet*, "one,"

[1] See examples below under "Subtractive Principle."

wintun, "Indian"; Tuolomne: *renge mewoom*, "one man"; Maidu: *kom maiduk*, from *maidu*, "Indian," and *kom*, possibly "whole"; Shasta: *tsec*, from *tsec*, "man"; Tlingit: *tlekha*, from *tle*, "one," *hka*, "man."

(*d*) *Extent and distribution.* Clear linguistic digital evidence similar to the examples given above has been found in about 40 per cent. of the languages examined, uniformly distributed over the continent. Doubtless further study will reveal similar evidence in other languages especially where adequate vocabularies have not been available.

(*e*) *Non-digital evidence.* We turn now to a consideration of the evidence that the origin of number words was non-digital in some languages. There are four phases to be considered. (1) *First Four Numerals.* The concepts of unity and duality are so fundamental that in many instances we may be sure they were named before formal finger counting gave names to the corresponding words. *One* has a connection with the first personal pronoun in some languages. *Two* seems often to come from roots denoting separation, "that" as distinguished from "this," or from ideas of pairs, being frequently related to the words for hands, feet, eyes, wings, husband and wife. *Three* is more frequently digital, but it seems sometimes to have a meaning of "more," "many," a plural as distinguished from a dual. Compare Micmac: *tchicht*, "three," with the cognate Delaware *tchitch*," still more." *Four* is sometimes expressed by a word meaning "complete," "right," "perfect." Its frequency as a sacred number among the North American Indians and its use in some cases as the base of a quaternary system[1] indicate that it is a unique word of non-digital origin. (2) *Arithmetical Operations.* Numbers higher than ten and in many cases those higher than five are expressed by arithmetical operations, and the digital meaning, even if present in the beginning, usually sinks into the background. The process of such combination begins earlier than the English in many Indian languages. We have numerous examples of $3 = 2 + 1$, $4 = 2 \times 2$, $4 = 2 + 2$, $6 = 3 \times 2$, $8 = 4 \times 2$, $10 = 5 \times 2$, $12 = 6 \times 2$, $9 = 3 \times 3$ and other rarer combinations. Thus there are many cases in which words for numerals above three are derived by purely arithmetical processes.[2] Of course there are the higher numerals, hundred, thousand, million, where they exist, in which we should rarely expect digital evidence. (3) *Marks of Completion.* When the Indian has counted ten or twenty he may use some reminder of the fact, such as a pebble, stick, arrow, grain of corn, etc. For example, Huchnom: 20, *pualya*, "one-stick-stand" and similarly for 40 and 60; in the same language 100 is *pual*, "one-stick" and similarly for 200; Maidu: 20, *penim nokom* "two arrow"; Gallinomero: 100, *tcacuto-hai*, "ten-stick." (4) *Superlative and Indefinite.* When a simple arithmetical combination is not used, especially for the expression of higher units, a superlative principle is sometimes found. Hundred is often expressed, "big ten" and thousand as "old hundred," "big hundred" or "too many to count"; e. g., Delaware: 1,000, *ngutti kittapachki*, "the great hundred"; Choctaw:

[1] See below under "Quaternary Systems."

[2] See examples below under "Duplicative Principle," "Ternary Systems."

1,000, *tahlepa siponki*, "old hundred"; Kwakiutl: 1,000,000 *tlinhi*, "number which cannot be counted"; compare the Greek "myriad."

We conclude that in North American Indian languages it is by no means true that number words, even as far as ten, always "gehen aus von den Fingern" although they probably do in a large majority of cases and the close connection can be traced in many instances. There is little uniformity as to method of formation, considerable diversity being found even in adjacent languages of the same family. This would indicate that their separation into tribes preceded the development of formal counting.

2. **Additive Principle.** Cantor says that addition and multiplication are two methods of counting as old as the formation of number words.[1] The additive principle is found of course in all numeral systems. Three phases of it are of interest in the American Indian languages.

(*a*) *Repetition.* This is the simplest form of the additive principle. If "one" is given, either as a symbol or as a word, "two" may be expressed "one-one," "three" as "one-one-one," etc., or by symbols as in the Roman numerals from one to four. In the gesture language of the Indians this is the method used, the fingers being the counters. In spoken language no instance has been found of "two" as "one-one," but there are several of "four" as "two-two"; e. g., Catawba: 2, *purra*, 4, *purrapurra*. In the Indian pictographs or hieroglyphics the simple repetition of strokes or notches is used, even for numbers up to a hundred. Sometimes these are grouped into tens by longer strokes or larger notches.

(*b*) *Addition to a Base.* The English does not begin to use the additive principle until ten is reached, but many Indian languages begin much earlier. The earliest instance found is in the Coahuiltecan: 1, *pil*, 2, *ajtic*, 3, *ajtic-pil*. In other languages we find such expressions as "6–2 added" for 8, "8 + 1" for 9, "12 + 3" for 15, and of course very often "5 + 1, 2, 3, 4," for 6, 7, 8, 9 and "10 + 1, 2, ··· 9" as in English for 11, 12 ··· 19. Those from 15 to 19 are also represented by "15 + 1, 2, 3, 4." An interesting variation is shown by the Maidu numerals from 16 to 19 which in translation are "one with man" for 16, "two with man" for 17 and similarly for 18 and 19 to 20 "man," the thought being "15 and one more on toward entire man."

(*c*) *Precedence.* Hankel and Fink call attention to a general law by which the written representation of numbers, when not confined to the mere rudiments, shows a tendency for higher numerals to precede the lower to represent addition.[2] Is there a similar tendency for the spoken order of numerals among the Indian languages? Does the lower precede the higher or vice versa? In about 250 languages sufficient facts were available for study of the method of formation of compound numerals by the additive principle. The groups from 5 to 10, from 10 to 20, and above 20 have been considered separately. Using the notation "$G + L$" to indicate "the greater is followed by the less" and "$L + G$" for the

[1] Cantor: *op. cit.*, p. 8.

[2] Hankel, H.: *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 32; Fink, K.: *History of Mathematics* (Beman and Smith trans.), Chicago, 1900, p. 8.

opposite, "5 + 1," "1 + 5," etc., to stand for pure number combinations in the order given, and "1 + *X*," "*X* + 1" for the indefinite cases of an unknown element (probably non-numerical such as "again," "besides," etc.) combined with the known numbers, we may summarize the results of an examination of these languages as follows.

In the 5–10 group for the pure number combinations, $L + G$ and $G + L$ occur with about equal frequency. But if we include the unknown compounds $X + 1$ and $1 + X$ with $G + L$ and $L + G$ respectively, $L + G$ predominates about 2 : 1. In the 10–20 group for pure number combinations $G + L$ predominates strongly, 8 : 1, a reversal of the order of the first group. But for $X + 1$ and $1 + X$, $L + G$ predominates slightly, the ratio being 4 : 3, so that $G + L$ predominates in the group 2 : 1. In the group of higher combinations $G + L$ predominates 16 : 1. Combining these results $G + L$ predominates altogether about 2 : 1. If the indefinite cases of $1 + X$ and $X + 1$ are excluded, only pure number combinations remaining, we approach close to a definite law. $G + L$ then predominates about 8: 1. $G + L$ and $L + G$ both occur in the same language in fully half the cases examined. In this particular a marked contrast with the multiplicative principle will be observed. It may be mentioned that our own (oral) system is mixed, $L + G$ from 10 to 20, and $G + L$ for higher numbers.

3. **Subtractive Principle.** Fink says: "In the verbal formation of a number system very rarely does subtraction come into use."[1] This statement is scarcely warranted for the systems of the American Indians for the principle has been found in 40 per cent. of the languages examined. It occurs most frequently in expressions for "nine"; e. g., "one finger left," "one from ten," "one from finished," etc., but also in the cases of 4, 7, 8, 14, 17, 18, 19 and the odd tens, 30, 50, etc. It is widely but not uniformly distributed over the continent. It occurs most frequently in the northern, eastern and central sections of the continent, less on the Pacific coast and least of all about the Gulf of Mexico. As is to be expected, "one subtracted" is most usual, occurring in about 30 per cent. of the languages, "two subtracted" in about 5 per cent. and "three subtracted" and "ten subtracted" in about 2 per cent. each. The use of the principle is found in some languages but not in closely related ones of the same family.

(*a*) *One Subtracted.* As examples we may give for *nine*, Unalit: *keka-mitatet*, "nearly ten," or *payuk-ostau*, from *payuk*, "one," *ostau*, "less?"; Uinta: *suromatampsuin*, "near ten"; Haida: *klath-skwanson*, from *klath*, "one," and *skwansin*, "ten." For *four* we have Takhtam: *roatcham*, from *mah-atcham*, "five"; Zuni: *awiten*, "all fingers all but done." For *fourteen*, Point Barrow: *akimiaxotaityuna*, "I have not quite 15." For *nineteen*, Alaska: *enuenok-otalia*, "twenty less one." *Thirty-nine*, Kulanapo: *pitikunanu-akhokaki*, "forty, one not." Arikara expresses both 7 and 9 from 8 and 10 respectively by means of a diminutive particle, "little ten," "little eight."

[1] Fink: *op. cit.*, p. 8.

(*b*) *Two Subtracted.* Onondaga: 8, *teg-ueron,* from *tegni,* "2"; Crow: 8, *nupa-pik,* from *nupa,* "2," and *pirake,* "10"; Kwakiutl: *matl-gwanatl,* from *matl,* "2."

(*c*) *Three Subtracted.* This occurs but rarely. Pawnee is noteworthy. In it, 17, 18, 19 are *tauit-kaki, pitkus-kaki, usku-kaki* from *usku,* "1," *pitkus,* "2," *tauit,* "3," and *kaki,* "less."

(*d*) *Ten Subtracted.* This is quite common among some of the California families, where the odd tens, 30, 50, 70, 90 are expressed as 40, 60, 80, 100 combined with ten, with the thought "60 lacking 10." In some cases however the thought may be "(40) and 10 on the way to 60," a variation of the additive principle not uncommon among these tribes [1]

4. **Multiplicative Principle.** This principle, like the additive, is of universal use in Indian as well as in all other languages for the formation of higher numerals. It seems to begin in the Indian languages with the expression of 4 as "2×2." It is constantly used in the formation of "secondary bases" in the ternary, quaternary, quinary decimal, and vigesimal systems. Our chief interest in studying this principle is in the question of precedence. A much more decisive law of precedence than in the case of the additive principle is found. About 200 languages afford data for the conclusion that the marked tendency in the formation of higher numerals by multiplication is for the lesser number to precede the greater. This coincides with both the oral and written English forms. In every group the type $L \times G$ predominates over the type $G \times L$ to a marked degree. For the formation of the tens, forms of the type 2×10 predominate over the type 10×2 in the ratio 5 : 1; for the hundreds, 2 : 1; for the thousands, 2 : 1; for others (e. g., $8 = 2 \times 4$), 17 : 1. Or altogether a predominance of $L \times G$ 4 : 1. Almost all the instances of $G \times L$ are found in the languages about the Gulf of Mexico and in the single but large Siouan family of the plains. If the Siouan languages alone were left out of consideration the predominance of $L \times G$ would be about 8 : 1. In only five languages do we find $L \times G$ and $G \times L$ both occurring in the same system, a decided contrast to the results noted in the study of the additive principle.

5. **Duplicative Principle.** A striking feature of the Indian numeral systems is the frequency of occurrence of a duplicative or pairing principle. In some instances 6 is expressed as "2×3," "again 3," "3, 3," "threes" and similarly for 4, 8, 10, and even 12. The large number of natural pairs, such as the eyes, hands, arms, wings, etc., suggests that counting by pairs might be the course of evolution followed by some languages. The standard historians of mathematics make little reference to such a principle as of any importance in the numeral systems of primitive people.[2] We do find Hankel, however, making the rather surprising statement in discussing the number words of uncivilized people, that ten is never expressed as two times five, but always by a simple number word.[3] This is not the case in American Indian languages; e. g., Gabrieleno: *wehes-mahar,* from *wehe,* "2," and *mahar,* "5"; Serranos: *wor-maharte,* from *maharte,* "5,"

[1] See example of Maidu given under "Additive Principle."

[2] Cantor: *op. cit.,* p. 5. [3] Hankel: *op. cit.,* p. 20.

wor, "2"; Patwin: *pampa-semta*, from *eti-semta*, "5," *pampet*, "2." Examples of its use for 4, 6, 8, 12 are Kutchin: 6, *neckh-kiethei*, 8, *nakhei-etanna*, from *nackhai*, "2," *kiethai*, "3," *etanna*, "4"; Kansas: 8, *kiya-tuba*, from *tuba*, "4," *kiya*, "again"; Shoshone: 4, *what-sowit*, from *what*, "2"; Cehiga: 12, *cape-nanba*, from *cape*, "6," *nanba*, "2"; Chwachamaju: 8, *kom-tca*, from *mitca*, "4," *ko*, "2."

In languages of the same family and even dialects of the same language there is variation in the use of this principle. And many examples can be given where the principle is not used at all in languages closely related to those in which it is. One feature is rather surprising; namely, of the approximately 125 cases of the probable use of this principle, it is far more common in the formation of four, six, eight than in the case of ten, even though ten is represented so commonly by the two hands. Fifty instances of its use for the formation of "8" are found, thirty-five for "4" (although some of these are somewhat obscure), twenty-five for "6," only ten for "10" and two for "12." Several languages seem to use it regularly in the formation of the even numbers to ten.

6. **Divisive Principle.** Historians of mathematics agree on the rarity of the use of this principle in the formation of numerals the world over.[1] Only two or three possible instances have been found in Indian languages. Thus we have Unalit: 10, *kolin*, and the literal meaning of the word, "upper half of the body," Point Barrow: 10, *kodlin*, "upper part (of body)," and similarly in other Eskimo languages. One other example has been found, Pawnee: 5, *sihuks*, "hands half" from *ishu*, "hand," and *huks*, "half," i. e., "half of the two hands."

7. **Fractions.** Although many tribes had numeral systems of integers running into the thousands and even to millions,[2] very few of them had any idea of fractions. Where we do find such ideas they are of the most rudimentary sort. "One half" occurs most frequently, but only in about a dozen cases as far as noted, while examples of the other fractions are almost negligible.[3] It is worth mentioning that the few instances we find are all of "unit fractions."[4] Onondaga shows the best development of fractions, but how meager for a language whose numerals are given to one million. Its fractions are: $\frac{1}{2}$, *sat wachenonk*, meaning uncertain; $\frac{1}{3}$, *achen-na-degayagui*, from *achen*, "3," meaning "thrice divided"; $\frac{1}{4}$, *gayeri-degayagui*, from *gayeri*, "4," "four times divided."

8. **Notation.** As already mentioned, few symbols for number other than the spoken words are found. But on grave posts, buffalo robes, tattooing and other mnenomic pictographs there are a few pictured symbols.[5] The usual method used for indicating numbers is by the repetition of single strokes, i. e., the additive

[1] Fink: *op. cit.*, p. 8; Hankel: *op. cit.*, p. 21.

[2] As to actual use of these numbers, see below, "Limits in Use."

[3] Some ethnologists state that words for the simpler fractions were common, but that they have been recorded but rarely by students of Indian languages.

[4] The pedagogical suggestion is worth noting. The use of integers developed long before fractions, and when fractions are introduced the unit fractions, i. e., those which have unity for the numerator, are the first to appear. In the Ahmes papyrus, one of the earliest known mathematical documents (Egyptian), a table is given for reducing common fractions to unit fractions, which were regarded as the standard type. See Cajori: *History of Mathematics*, New York, 1894, p. 14; Ball: *Short History of Mathematics*, London, 1893, p. 3; Cantor, *op. cit.*, Chap. I.

[5] For reproductions of all those known in North America and discussion of them see articles by Mallery in 4th and 10th Ann. Rpts. Bur. Eth., Washington.

principle in its simplest form. Sometimes the strokes are arranged in rows of ten or every tenth stroke is made longer than the others. Instances of this kind are found for the expression of numbers as high as thirty. They are found among the more highly civilized Indians of the middle west, especially among the Dakotas. On the other hand the Comanches are said to have been "ignorant of the elements of figures, even of a perpendicular stroke for one."[1] Pure numeral notation is not always found. Frequently the number of objects is expressed by the repetition of the symbol for the object the desired number of times, especially in the case of men or tepees. Another method is by dots. A man's head with eight dots above it in one case means nine men, the head itself counting for one. Another picture gives a head over which are thirty black dots in three lines of ten each. This is said to mean thirty men, not thirty-one. Thus the usage is not fixed. The use of notches cut on sticks was frequent, not only in the middle west but in California and on Puget Sound in western Washington. At the San Gabriel mission in California every tenth notch was cut entirely across the stick instead of only in the corner.

(*To be continued.*)

SYNTHETIC PROJECTIVE GEOMETRY AS AN UNDERGRADUATE STUDY.

By W. H. BUSSEY, University of Minnesota.

Modern Geometry.—Synthetic projective geometry is called modern synthetic geometry because the theories and methods that make of its propositions a homogeneous and harmonious whole are due for the most part to men of a time near our own, namely Poncelet, Möbius, Steiner, Chasles, Von Staudt, Reye, Cremona, and others. Some of its theorems can be traced back to Euclid, Apollonius, and Pappus, but it is in the work of Pascal and Desargues that the germs of the modern theories are to be found. When he was only sixteen years old Pascal discovered his famous theorem about a hexagon inscribed in a conic and made it the basis of a treatise on conics which is unfortunately lost. It was never published but we know something of its contents and extent from a report made by Leibnitz who saw it in Paris. Two of the most important theorems of projective geometry go by the name of Desargues. One is that if any two triangles in a plane or in space are such that corresponding vertices are collinear in pairs with a fixed point, then the corresponding sides are concurrent in pairs with a fixed straight line. The other is that any transversal meets a conic and the pairs of opposite sides of any inscribed quadrangle in points of an involution. But unfortunately for the development of the subject the work of Pascal and Desargues was overshadowed by the geometry of their contemporary Descartes. Analytic geometry and the calculus, invented about thirty years later, became the

[1] Eakins, D. W.: in Schoolcraft's *Indian Tribes*, Philadelphia, 1851–60, Vol. 1, p. 416.

chief instruments of progress in geometry as in all mathematics, and the work of Pascal and Desargues lay undeveloped for nearly 200 years. The revival of synthetic geometry by methods similar to theirs was due chiefly to Poncelet, and for that reason the origin of projective geometry is usually ascribed to him.

Synthetic versus Analytic Geometry.—Synthetic projective geometry is still overshadowed by analytic geometry and the calculus and it does not have the place in the undergraduate curriculum that it ought to have. Indeed it may be said that it has no recognized place, for in most American colleges and universities it is not offered at all, and there is nothing like uniformity in the conditions under which it is offered in those that do give a course in the subject. In some colleges it is primarily for graduates and in others it is primarily for undergraduates. More often it is for both undergraduates and graduates. The subject is offered three hours a week for a semester, two hours a week for a year, and three hours a week for a year. A number of colleges are offering an introductory course as an elective to juniors and seniors who have had a first course in the calculus. In a few cases the only prerequisite is a first course in analytic geometry and the course is open to sophomores. When given under these conditions the course must of necessity be very different from that given in some institutions to graduate students who have a reading knowledge of French and German. At Bryn Mawr a lecture course in the subject is offered as a post-major course to juniors and seniors who have taken college mathematics five times a week for two years, and a reading knowledge of French and German is a prerequisite.

The subject if properly taught is well within the grasp of the undergraduate who has had a first course in the calculus. An introductory course of at least three hours a week for a semester should be offered to such students in every college, and those preparing to teach mathematics in the high school should be encouraged to take it. The subject is so recommended at Vassar, the prerequisite being analytic geometry. It is a significant fact that most of the prominent colleges for women offer courses in synthetic projective geometry to junior and senior undergraduates. Bryn Mawr, Mount Holyoke, Smith, Vassar, Wellesley, Wells, and Goucher College of Baltimore do so. The course at Bryn Mawr requires considerable mathematical maturity and a reading knowledge of French and German, but at the other colleges the course is less advanced. At two of them the prerequisite is analytic geometry.

Curriculum Now Planned for the Prospective Graduate Student.—The reason why projective geometry has no well-defined place in the undergraduate curriculum is not hard to find. The curriculum is planned primarily for those who are to go on with graduate work in mathematics or physics. It takes two years, sometimes two and a half, for the student to get through college algebra, trigonometry, analytic geometry, and a year of the calculus. Then if he is to be well prepared for graduate work he ought to take advanced calculus, solid analytic geometry, and a course in determinants and the theory of equations; a second course in plane analytic geometry is needed for a working mastery of the subject,

and he ought to have at least two years of undergraduate physics and one of astronomy, and perhaps a year of theoretical mechanics. He is not likely to have time for any more electives. In fact most graduate students have not had as good a preparation as this. One or more of these subjects is likely to be left for the first graduate year. But every one of the subjects is of more importance to the prospective graduate student than synthetic projective geometry, because they are all on the main line of progress in mathematics and are all prerequisites for advanced work in applied mathematics. Projective geometry is more an end in itself. It is not needed so much as a prerequisite for something else. It is not a prerequisite for graduate work in mathematical physics or astronomy. So it is left out by most prospective graduate students even if they have an opportunity to get it as undergraduates, as most of them do not. They get it as graduate students or they do not get it at all. Then they are more mature and the course can be made, and usually is made, to cover much more ground and to be more intensive than an undergraduate introductory course in the subject. A reading knowledge of French and German can be assumed, and there need be no hesitancy in making the course one in pure geometry. It can be as abstract as the teacher pleases and it ought to involve to a greater or less extent a study of the foundations of geometry. Properly qualified seniors could take such a course, but it is not suitable for those who have just completed a first course in the calculus.

Part of a Liberal Education.—The students who take elective courses in mathematics after having had a first course in the calculus are of three kinds; those who are preparing to do graduate work in mathematics, physics, or astronomy; those who are going to college for a liberal education and who are taking mathematics because they like it and not because they expect to use it; and those who are taking mathematics in college as a preparation for teaching mathematics in the high school. For reasons already given a student of the first kind can afford to wait and take projective geometry when he is a graduate. If he is interested primarily in physics or astronomy, he may not want it at all. A student of the second kind is prepared by his one year's work in the calculus for the undergraduate courses in physics and astronomy that he may wish to take. Any other courses in mathematics that he elects he will choose on account of their interest to him. He may be one of those who always liked geometry better than algebra, and there are many such. If he belongs to this class, a course in modern synthetic geometry would probably be more interesting to him than any other course that could be offered. At any rate he ought to have a chance to take it if he wants to.

Preparation for Teaching.—These remarks about the student who is taking mathematics as a part of a liberal education apply equally well to the student of the third kind who is taking mathematics as a preparation for teaching. In many institutions this class is the largest of the three. In coeducational colleges it consists mostly of young women. College algebra, trigonometry, analytic geometry, and the calculus are a better preparation for teaching algebra than

they are for teaching geometry. They involve almost constant algebraic manipulation even when their subject matter is not algebra. It is often said that a student is not really a master of high school algebra until he has been through a course in the calculus. Many prospective high school teachers take, in addition to these subjects, a course in determinants and the theory of equations and thereby make their preparation for teaching algebra so much the better. But what preparation for teaching geometry does the prospective high school teacher get that is at all comparable with the preparation for teaching algebra afforded by these courses? Of course all courses in mathematics are indirectly good preparation for teaching geometry. Experience in one branch of mathematics counts in the study of any other branch to some extent, but that does not alter the fact that, as it now stands, the undergraduate curriculum in mathematics affords a better preparation for teaching algebra than geometry. Algebra and analysis have the right of way. The geometry of the high school is synthetic. It consists largely of memory work and deductive reasoning. It holds its place in the high school not only because it is a useful subject in the work of the world but also because it is a good course in logical reasoning. What better preparation for teaching it can the student get than a course in synthetic projective geometry? It is more closely related to the geometry of Euclid than any other branch of mathematics. It gets the student back to synthetic methods. It involves more of reasoning and less of the technique of manipulation than either analytic geometry or the calculus, and it can be taught so as to be as easily within the mental grasp of the college sophomore as either of these subjects. Construction problems with the ruler and compass, with the ruler alone, with the graded ruler, and with the parallel ruler should be an important part of the introductory undergraduate course in synthetic projective geometry. There is a wealth of such problems in Cremona's *Elements of Projective Geometry*, and the teacher can find more, if they are wanted, in Adler's *Geometrische Konstruktionen*. To draw a line through the inaccessible point of intersection of two given lines by means of the ruler alone, to draw a line through a given point and parallel to a given line by means of the graded ruler or the parallel ruler, and the many other constructions possible by the principles of projective geometry are fascinating to lovers of geometry, and the constructions that can be made by means of the theorems of Pascal, Brianchon, and Desargues are a revelation to the student. What better preparation is there than this for teaching that part of the high school course which deals with construction problems?

Generality and Power of Synthetic Geometry.—Many a student leaves college to become a teacher of high school geometry with the notion that no progress in geometry is possible except by means of coördinates and algebra, and that there is no higher geometry more closely related to the geometry of Euclid. This ought not to be so. The course in modern synthetic geometry would not be so interesting to the student nor so valuable in broadening his mental horizon if it were merely a sequel to Euclid. It is more than that, inasmuch as it is characterized by the great generality and power of its methods and theorems. It is

said that Pascal, in his lost treatise on conics, had 400 corollaries to his famous theorem of the mystic hexagram. The method of projection whereby theorems about circles are translated into theorems about conics; the principle of duality whereby every theorem about the incidence of points, lines, and planes yields another about planes, lines, and points; and the principle of continuity, which Chasles prefers to call the principle of contingent relations,—these are very general in their applications and are of great power in research. There is nothing to match them in the geometry of the Greeks. Their geometry consists for the most part of isolated theorems. They were more anxious to convince than to explain, and the student gets no information about the way in which the various theorems were discovered. In modern synthetic geometry, on the other hand, the student is to a considerable extent led to understand how many of the theorems came to be discovered. A good example of the great power of the principle of continuity is afforded by the long list of deductions that can be made from the theorems of Pascal and Brianchon. The student can discover some of them for himself as soon as he is let into the secret of the method. The principle of continuity indicates what the theorems are. Afterwards they can be proved rigorously. This modern synthetic geometry is very different from the synthetic geometry of the Greeks both in subject matter and in method, but it has enough in common with it to make it a good preparation for the teaching of high school geometry.

One Semester for the First Course.—The college student who is preparing to be a high school teacher must be qualified to teach more than one subject, at least in the beginning of his teaching career. Teachers in small high schools sometimes have to teach four or five. Certain courses in psychology and education are required for state teachers' certificates, and it is expected that every high school teacher will have a good liberal education. The student who is going to be a teacher with mathematics as a specialty ought to take courses in college physics and astronomy. For these reasons the prospective teacher often finds that he cannot take more than one or two half year courses in mathematics after he has had a year of the calculus. In many colleges the second half year of the calculus is normally taken in the first half of the student's junior year. Synthetic projective geometry ought to be available for these people, many of whom will be good high school teachers of geometry although they would not all, or even nearly all, be successful in graduate work in mathematics. But the course should not be so arranged that the student must take a whole year of the subject. It should be a half year course, or if a year's work is offered it should be so arranged that the work of the first semester constitutes a good course by itself, just as the first course in analytic geometry is one that is worth while even if the student never takes any more of it. A good introduction to projective geometry can be given in a half year, as in the case of analytic geometry. Then the second half year's work can be thought of as a second course for those who want it and have time to take it. It will be on a par with the second course in analytic geometry as far as its relation to the first course is concerned. In some

colleges a half year of synthetic projective geometry is followed by a half year of advanced analytic geometry, or vice-versa, neither being a prerequisite for the other. Sometimes the two are offered in alternate years so that the student who wishes can get either or both. This plan is necessary in some colleges because there is not enough demand for both courses every year. Another plan is to give a year's course in modern geometry, synthetic and analytic methods both being used all the way through the course. This makes a good year's course but is not so well adapted to the needs of the student who can take only one semester's work. This plan is followed in a number of colleges, but many times the course announced as modern geometry, whether it is a year or a half year course, is too exclusively taken up with the analytic point of view at the expense of synthetic projective geometry. This is due to some extent to the fact that many college teachers of mathematics have never had an adequate training in modern synthetic geometry, and consequently their interest is on the analytic side. This is another consequence of the fact that projective geometry is not on the main line of progress in mathematics. Thus the present status of modern synthetic geometry tends to perpetuate itself.

A Brief Outline of a Suitable Course.—The introductory course of a half year three times a week, or its equivalent, should not be a course in pure geometry that avoids the measurement of geometric magnitudes and all metric properties, like the geometry of Von Staudt. Nor should it be a study of the foundations of geometry, or involve such a study, to any considerable extent. The plane geometry of the high school should be assumed in its entirety just as it is for the first course in analytic geometry. The course will from its very nature deal largely with projective rather than metric properties of geometrical figures, but to avoid all metric notions is not wise in a course open to college juniors. Anharmonic ratios should be used freely, and the measurement of geometric magnitudes is involved in their definition.

The following is a brief outline of a course that can be completed in a half year, if not too much time is given to any one of the topics. The author of this paper has tried it with satisfactory results:

> Central projection and figures in perspective, points at infinity, the principle of duality, projective geometric forms, harmonic forms, anharmonic ratios, the construction of geometric forms, projective forms in relation to the circle, projective forms in relation to the conic sections, the theorems of Pascal and Brianchon and their many corollaries, the principle of continuity, projective ranges on a conic and projective series of tangents to a conic, superposed projective ranges and involutions, the theorem of Desargues about a conic and an inscribed quadrangle, problems of the second degree, pole and polar, centers and diameters. Construction problems to be given all the way through the course, as many as there is time for.

The Question of a Suitable Text.—There is no satisfactory textbook for this course. Cremona's *Elements of Projective Geometry* can be used to advantage if it is supplemented to a considerable extent by lectures, and if good judgment is used in omitting some paragraphs and in substituting for others. The writer has used the book in this way but has found it necessary at times to substitute other proofs for the ones given by Cremona because his proofs involved para-

graphs that had been omitted. Or the course can be made a lecture course with considerable work in assigned problems and outside reading. Reye's *Geometry of Position* (volume 1, English translation by T. F. Holgate) is out of print, but it is likely to be in the library so that it can be used for reference. The *Projective Geometry* by Veblen and Young is too advanced and abstract for students who have had only two years of college mathematics, and besides it is not suitable for a short course. But it can be used to advantage for reference at times. Emch's *Projective Geometry* and Miss Scott's *Modern Geometry* get at the subject mostly from the side of analytic geometry, but they can be used for reference at times. There are some English books that the teacher may want to have in the library for occasional reference, although no one of them is suitable for use as a textbook for the course, namely, Russell's *Elementary Treatise on Pure Geometry*, Milne's *Elementary Treatise on Cross Ratio Geometry*, Filon's *Projective Geometry*, Pickford's *Elements of Projective Geometry*, and Durell's *Plane Geometry for Advanced Students*.

It is to be hoped that some one will publish a book on synthetic projective geometry that will be suitable for use as a textbook for an introductory course in the subject. In the meantime the course if given will have to be conducted in one of the ways suggested above.

A NOTE ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

By C. R. MAC INNES, Princeton University.

In showing how to solve linear differential equations with constant coefficients, the usual way of finding the complementary function seems to be to guess at the various parts and then to prove that the guesses are correct. A much more satisfactory way is to define the differential operators $D - \alpha$, $D - \beta$, etc., to show that they obey the laws of addition and multiplication, and then to proceed in a straightforward way to solve the problem.

Having shown that $\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$ may be written as $(D - \alpha)(D - \beta)y = 0$, one sees that a solution of $(D - \beta)y = 0$ is a solution of the original equation since $(D - \alpha)0 \equiv 0$. Since the factors are commutative any one may come last and we can get all the parts of the complementary function except those arising from repeated factors.

If $(D - \alpha)(D - \beta)y = X$, put $(D - \beta)y = v$; then $(D - \alpha)v = X$. Integrating this in the usual way one gets

$$v = (D - \beta)y = e^{\alpha x}\int e^{-\alpha x}X dx.$$

Hence the rule: *To remove a factor $(D - \alpha)$ from the left side, multiply the right member by $e^{-\alpha x}dx$, integrate and multiply by $e^{\alpha x}$.*

If we have $(D-\alpha)^2 y = 0$, we may therefore get $(D-\alpha)y = c_1 e^{\alpha x}$. On repeating this operation,

$$y = e^{\alpha x} \int e^{-\alpha x} c_1 e^{\alpha x} dx = c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

If we have $(D-\alpha)^3 y = 0$, we get in precisely the same way

$$y = c_2 x^2 e^{\alpha x} + c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

The complementary function would then be completely known on proving the usual theorem that if $v_1, v_2, \cdots v_n$ are solutions of $f(D)y = 0$ then $c_1 v_1 + c_2 v_2 + \cdots c_n v_n$ is also a solution.

One need hardly draw attention to the fact that the complete value of y will be obtained if the factors of

$$(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)y = X$$

are removed in turn by the above rule, the constant of integration being introduced each time.

A GRAPHICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF THE FIRST ORDER.*

By T. R. RUNNING, University of Michigan.

The following graphical solution of the differential equation of the first order was derived while trying to solve a problem involving two such equations. As an illustration, these equations will be solved after a discussion of the method of solution has been given.

If the primitive of a differential equation which expresses the condition of a problem can not be found, or can be found with difficulty, it may be desirable to have a method of approximate solution which does not involve a great deal of labor. It is thought that the method given is more easily carried out than any other which has come to the writer's attention.[1]

The solution depends upon the simple fact that the area under the derived curve is equal to the corresponding ordinate of the integral curve. In Fig. 1

$$\text{Area } OPQ = RQ.$$

If in the equation

$$\frac{dy}{dx} = f(x, y)$$

we assign particular values to y and plot the resulting equations with dy/dx as

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* First part of a paper presented to the American Mathematical Society at Madison, September 9, 1913.

[1] For a comparison of approximate methods see "Beitrag zur näherungsweisen Integration totaler Differentialgleichungen," by W. Kutta, *Zeitschrift für Mathematik und Physik*, Vol. 46, pp. 435–453.

ordinate and x as abscissa, we shall have, designating each curve by the corresponding value of y, a family of curves as shown in Fig. 2.

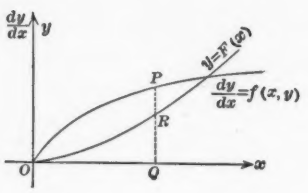

FIG. 1.

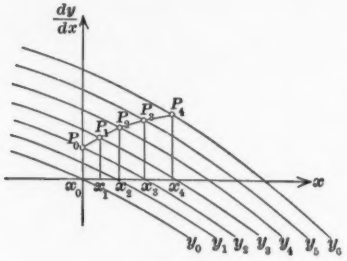

FIG. 2.

Suppose the initial condition to be $y = y_2$ when $x = 0$. Draw a curve, which for an approximation may be a straight line, from P_0, where the curve y_2 crosses the axis of dy/dx, to the curve y_3, such that the area under it shall be equal to $y_3 - y_2$. From this point, P_1, draw a line to a point P_2 on the curve y_4 such that the area under the line P_1P_2 shall be equal to $y_4 - y_3$. This process continued will give any desired number of points on the integral curve. The curve connecting the points $P_0P_1 \cdots P_n$ is the derived curve.

It is to be noted that the integral curve is not obtained by graphically integrating the derived curve, but the coördinates of its points are read from the figure. Thus (x_0, y_2), (x_1, y_3), (x_2, y_4), (x_3, y_5), etc., are points which determine the integral curve. If a close approximation is required the values of y must be so chosen that the corresponding curves will lie near together.

The problem mentioned was one which gave rise to the two differential equations

$$\frac{dy}{dx} = \frac{1800x - 132y^{\frac{1}{2}}}{11y^{\frac{1}{2}} + 1000}, \tag{1}$$

$$\frac{dy}{dx} = \frac{1740 - 240x - 132y^{\frac{1}{2}}}{11y^{\frac{1}{2}} + 1000}. \tag{2}$$

It was required to find the maximum value of y and the corresponding value of x, knowing that when $x = 0$, $y = 0$, equation (1) was to be satisfied between $x = 0$ and $x = \frac{29}{34}$, equation (2) between $x = \frac{29}{34}$ and $x = 7\frac{1}{4}$. There was another equation to be satisfied beyond $x = 7\frac{1}{4}$ but, as the conditions clearly showed, the maximum value of y occurs before x reaches the value $7\frac{1}{4}$, it need not be considered.

The curves which correspond to different values of y are straight lines and shown in Fig. 3.

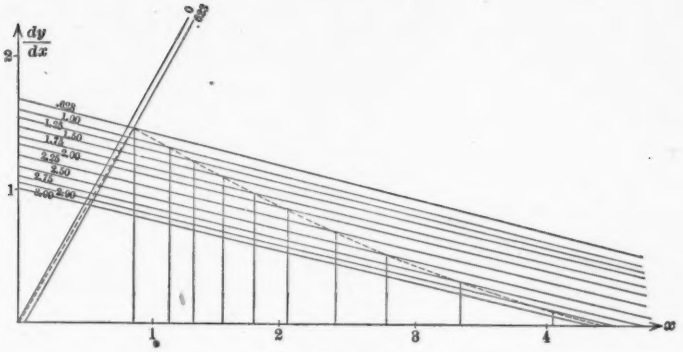

FIG. 3.

y will be a maximum when $dy/dx = 0$. This value is 2.926 and the corresponding value of x is 4.492. The values chosen for y are indicated on the figure. It is seen that when $x = \frac{29}{34}$, $y = .623$. This value must, therefore, be used in both differential equations.

The method was applied to the problem given in Kutta's article, above referred to, for comparing the five different approximations. With squared paper ruled to twentieths of an inch and using two inches as the unit, the approximation was closer than that of any one of the first four but not quite so close as Kutta's, while the work necessary to obtain the result was much less.

In a later paper it is proposed to extend the method to the solution of the differential equation of the second order.

BOOK REVIEWS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

Practical Mathematics. By CLAUDE IRWIN PALMER, Assistant Professor of Mathematics in the Armour Institute of Technology. The McGraw-Hill Book Company, New York, 1913. Four volumes. $0.75 each.

We have here in four small, neatly bound and clean cut volumes a very interesting presentation of elementary mathematical subjects from Arithmetic through Trigonometry. The work has been prepared for persons of maturity, being, as stated by the author, the outgrowth of a course in practical mathematics given in the evening classes of the Armour Institute of Technology.

The titles of the four little volumes are:

Part I. Arithmetic with Applications (150 pages).

Part II. Geometry with Applications (166 pages).

Part III. Algebra with Applications (189 pages).

Part IV. Trigonometry and Logarithms (160 pages).

These volumes are admirably adapted to meet the needs of the persons to whom they are addressed. The treatment throughout is simple and concise; demonstrations are largely avoided; and the whole work abounds in interesting practical exercises drawn from a large variety of sources. These exercises serve to make the books of great value to all teachers of elementary mathematics as a source of fresh and instructive problems.

W. C. BRENKE.

Computing Tables and Mathematical Formulas, arranged for the use of high schools and colleges. By E. H. BARKER. Ginn and Company, Boston, 1913. Narrow 12 mo, semi-flexible cloth, v+88 pages. $0.75.

This admirable little book of tables, 4 inches by 7 inches, contains a wealth of information which furnishes material for many problems of ordinary computation besides being very convenient for problems involving logarithms. Fifteen tables present the ordinary trigonometric functions and logarithms as well as powers (squares and cubes), roots, circumferences, areas, for the numbers from 1 to 1,000, trigonometric formulas, volumes of spheres (1 to 100 in diameter), standard gauges, decimal equivalents of 32ds and 64ths, specific gravities, weight of a cubic foot of various materials, and convenient equivalents. Any high school pupil, and even the ordinary college student, would profit by having such a book placed in his hands. The objection to the work from this point of view, however, is the price. Compare this book of 88 pages at 75 cents with the *Vierstellige Tafeln* by Schubert-Haussner at 22 cents. The German work is in two colors and consists of 175 pages, being volume 81 of the well-known *Sammlung Göschen*, G. J. Göschen'sche Verlagshandlung, Berlin W. 10, Genthiner-Strasse 38. Under the new tariff law teachers will be able to import such works free of duty for their students and until similar tables are published at a more moderate price in this country the German tables are to be preferred. Aside from the price, even, it is highly desirable that our high school students should occasionally see and handle foreign books.

L. C. KARPINSKI.

An Introduction to Mathematics. By A. N. WHITEHEAD. Henry Holt and Co., New York, 1911. 256 pages. $0.75.

Working mathematicians have usually not taken the time and trouble to render their point of view, the nature of their subjects of interest, and their methods intelligible to a layman. This state of things has no doubt been brought about by a feeling widely prevalent that the layman, even though intelligent, is not able to understand these things without first passing through a considerable course of training. So far as results and details of work are concerned this feeling is certainly justified.

But it is at least open to question whether the really fundamental ideas, stripped of all technicality in verbiage and results, are not after all of such simple

character as to be capable of presentation to any intelligent reader who may be interested in knowing what they are and how they are related to life and thought.

If such a presentation could be made it would obviously be of no little advantage to the science of mathematics. Many of us must have felt our extreme isolation from our fellow-workers in other fields; and we would welcome anything which would properly serve to narrow the gap between us. One great means of scientific progress is in the interconnections and interrelations between separated domains of thought.

Again, all of us have noticed how often the study of mathematics in the college (and even in the high school) begins with disappointment. The ideas and processes to which the student is early introduced do not command his interest. May it not be true that we give too much attention to the trivialities of computation and too little to the fundamental principles, always interesting, on which the science is built?

It appears, then, that both the student and the general reader might find it of great value to have at hand a simple and straightforward, yet accurate, discussion of the fundamental ideas of the mathematical sciences. This is what the author of the little volume at present under review has given us in a lucid and interesting exposition. It augurs well for a dissemination of interest in mathematical questions when men of such ability as Dr. Whitehead are willing to take the time to induct others into a clear apprehension of the elementary and fundamental notions of the science.

The object of this book is not to teach mathematics, but to enable students from the very beginning of their course to know what the science is about, and why it is necessarily the foundation of exact thought as applied to natural phenomena. There is no technical mathematical detail to embarrass the non-mathematical reader; and yet even the student of mathematics will find in it many illuminating statements. The book deserves a wide circulation.

It can be recommended unhesitatingly to any layman who desires to know the simple conceptions which lie at the basis of modern mathematics and to ascertain how they have grown up during the progress of thought.

A general idea of the contents of the book may be gained from the following list of chapter headings: the abstract nature of mathematics, variables, methods of application, dynamics, the symbolism of mathematics, generalizations of number, imaginary numbers, coördinate geometry, conic sections, functions, periodicity in nature, trigonometry, series, the differential calculus, geometry, quantity.

R. D. CARMICHAEL.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

396. Proposed by H. E. TREFETHEN, Colby College.

Show that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

GEOMETRY.

424. Proposed by H. E. TREFETHEN, Colby College.

In a given triangle ABC, determine by geometric demonstration the point O so that the sum of the distances, $AO + BO + CO$, shall be a minimum.

425. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the ratio of the areas, A_1 and A_2, of the parabolas formed by projectiles whose ranges are the same and whose angles of projection are complements of each other.

CALCULUS.

345. Proposed by C. N. SCHMALL, New York City.

"Of all the quadrilaterals which can be formed from four given sides, that which is inscriptible in a circle has the greatest area."

[From GOURSAT-HEDRICK, *Math. Anal.*, p. 133, ex. 5.]

Proposer's Remark.—Prove this, and furthermore show that when only three sides are known, the length of the fourth side of the maximum quadrilateral is the root of a cubic equation.

346. Proposed by C. N. SCHMALL, New York City.

Given the height of an inclined plane, to find its length so that a given force, acting on a given mass in a direction parallel to the plane, may draw it up in the shortest time.

Note.—This question has a practical application in the case of a truck, of known height, upon which a mass is to be loaded by means of skids.

MECHANICS.

280. Proposed by C. N. SCHMALL, New York City.

Given the distance d between two smooth hooks in the same horizontal line. Show that the shortest string which can form a catenary, with these hooks for points of support, is de, where e is the base of the Naperian system of logarithms.

281. Proposed by C. N. SCHMALL, New York City.

ABC is a triangle inscribed in a circle, center O, and L, M, N, are the centers of gravity of the sectors AOB, BOC, COA. Show that

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3\pi.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

197. Proposed by E. T. BELL, Seattle, Wash.

Show that in the expansion of $\dfrac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1$, where p is a prime, the coefficients of the various powers of z are divisible by p. [EISENSTEIN, *Crelle, t. 27, p. 282.*]

198. Proposed by ARTEMAS MARTIN, Washington, D. C.

Prove that every even number is the sum of two prime numbers.

Note.—This problem has long been known and no proof has ever been given. [EDITORS.]

CORRECTION.—Number **192**, incorrectly given as **188** in the June issue, page 196, should have the zeros on the right of equations (1), (2), (3) each replaced by □, the symbol for a "square number."

SOLUTIONS OF PROBLEMS.

ALGEBRA.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{1}{6}, \cdots, u_1, u_2, u_3, \cdots$, show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n=\infty} = \frac{1}{2}.$$

SOLUTION BY H. A. LEVY, Houghton, Mich.

Consider the sequence $\frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \cdots, \frac{q-1}{q}$.

(1) Suppose q to be an odd prime. Then the quantities $\frac{n}{q}$ $(n = 1, 2, 3, \cdots, q-1)$ are irreducible fractions, and $q-1$ must be an even number, say $2k$. Then

$$\sum_{n=1}^{q-1} \frac{n}{q} = \frac{1+2+3+\cdots+(q-1)}{q} = \frac{q-1}{2} = k.$$

Hence, in such a sequence, there are $2k$ terms, and the sum of the terms is k.

(2) Suppose that q is an even number of the form $2^r p$, where p is any odd prime except one. All the fractions with even numerators are then reducible, and the sequence is

$$\frac{1}{2^r p}, \frac{3}{2^r p}, \frac{5}{2^r p}, \cdots, \frac{2^r p - 1}{2^r p}.$$

In this sequence, at least one term is reducible, viz., $\frac{p}{2^r p}$. If there are more, they are the terms $\frac{p}{2^r p}, \frac{3p}{2^r p}, \frac{5p}{2^r p}, \cdots$ up to the greatest odd multiple of p less than $2^r p$. This odd multiple must therefore be of the form $4k+1$. Hence an odd number of terms are reducible. Furthermore, the numerator of the last term of the sequence, $2^r p - 1$, is of the form $4k+1$, so that there is an odd number of terms in the sequence. Rejecting the reducible terms, therefore, leaves an even number of irreducible terms.

If $r = 1$, the number of terms is $p-1$, an even number. The sum of the terms is then

$$\left(\frac{1}{2p}+\frac{3}{2p}+\frac{5}{2p}+\cdots+\frac{2p-1}{2p}\right)-\frac{p}{2p}=\frac{p}{2}-\frac{1}{2}=\frac{p-1}{2}.$$

That is, the sum of the terms is half the number of terms. The extension of the proof for the case where $r > 1$ is obvious.

(3) Suppose q is the product of two primes, say $l \cdot m$. If either l or m is 2, this reduces to the case just considered. The sequence is now

$$\frac{1}{lm}, \frac{2}{lm}, \frac{3}{lm}, \cdots, \frac{lm-1}{lm},$$

an even number of terms, since lm is odd. An even number of terms are reducible. They are

$$\frac{l}{lm}, \frac{2l}{lm}, \frac{3l}{lm}, \cdots, \frac{kl}{lm}, \frac{m}{lm}, \frac{2m}{lm}, \frac{3m}{lm}, \cdots, \frac{k'l}{lm}.$$

k and k' are both even, because k is the greatest multiple of l less than m and k' is the greatest multiple of m less than l. This leaves an even number of irreducible fractions in the sequence.

The sequence then is

$$\frac{1}{lm}+\frac{2}{lm}+\frac{3}{lm}+\cdots+\frac{lm-1}{lm}$$
$$-\left[\frac{1}{m}+\frac{2}{m}+\frac{3}{m}+\cdots+\frac{m-1}{m}+\frac{1}{l}+\frac{2}{l}+\frac{3}{l}+\cdots+\frac{l-1}{l}\right].$$

There are $lm - l - m + 1$ terms in the sequence, and the sum of the terms is

$$\frac{lm-l-m+1}{2}.$$

(4) Hence in all cases, the number of irreducible fractions having the same denominator is even, and their sum is half the number of terms.

(5) There remains to be considered one exception, the fractions having 2 as a denominator. Here there is only one term, but the sum of the terms is still one half the number of terms.

(6) Associating together the sequences having 2 as a denominator, 3 as a denominator, 4 as a denominator, etc., we have, besides the term $\frac{1}{2}$,

a sequence of $2k'$ terms whose sum is k',
a sequence of $2k''$ terms whose sum is k'', etc.

Adding these, we have

$1 + 2(k' + k'' + k''' + \cdots)$ terms whose sum is $\frac{1}{2} + k' + k'' + k''' + \cdots$.

Therefore

$$\frac{\sum_1^m u_i}{n}=\frac{\frac{1}{2}+k'+k''+k'''+\cdots}{1+2(k'+k''+k'''+\cdots)}=\frac{1}{2}.$$

Also solved by H. C. FEEMSTER.

GEOMETRY.

415. Proposed by W. D. CAIRNS, Oberlin, Ohio.

For three points P_1, P_2 and P_3 on a straight line

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

$$P_2P_3 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2},$$

$$P_1P_3 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2},$$

while $P_1P_2 + P_2P_3 = P_1P_3$; prove directly that the sum of the first two radical expressions equals the third.

SOLUTION BY THE PROPOSER.

Introducing a parameter notation, the equation of the straight line $P_1P_2P_3$ may be written

$$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c} = \lambda,$$

or $x = x' + \lambda a$, $y = y' + \lambda b$, etc.; whence, for the three given points (as indicated by subscripts),

$$x_1 = x' + \lambda_1 a, \quad y_1 = y' + \lambda_1 b, \quad \text{etc.};$$

$$x_2 = x' + \lambda_2 a, \quad \text{etc.}, \quad \text{etc.};$$

$$x_3 = x' + \lambda_3 a, \quad \text{etc.}, \quad \text{etc.}$$

The substitution of this notation in the given radical expressions gives them the form

$$(\lambda_2 - \lambda_1)\sqrt{a^2 + b^2 + c^2},$$

$$(\lambda_3 - \lambda_2)\sqrt{a^2 + b^2 + c^2},$$

$$(\lambda_3 - \lambda_1)\sqrt{a^2 + b^2 + c^2}.$$

Whence the desired result is evident.

Also solved by H. C. FEEMSTER, R. M. MATHEWS, A. M. HARDING, JAMES A. BULLARD, and C. A. BARNHART.

416. Proposed by W. H. BUSSEY, University of Minnesota.

Let AB be a diameter of any circle, and let C be the third vertex of the equilateral triangle of which AB is one side. Divide AB into six equal parts and let P denote the second point of division from A (*i. e.*, $AP = \frac{1}{3}AB$). Draw the straight line CP and produce it to meet the circle in R. Prove that AR is one-sixth of the circumference.

SOLUTION BY E. L. BROWN, Denver, Colorado.

Let O be the center of circle. Draw OD perpendicular to CP. In right triangle COP, $PO = r/3$, $CO = r\sqrt{3}$, $CP = 2r/3\sqrt{7}$, $PD = r/6\sqrt{7}$, $CD = 9r/2\sqrt{7}$, $OD^2 = 3r^2/28$. In right triangle ODR, $DR = 5r/2\sqrt{7}$.

$$PR = DR - DP = r\sqrt{7}/3 = \tfrac{1}{2}CP.$$

But $PA = \frac{1}{2}PB$. Therefore triangles APR and CPB are similar. Hence $AR = \frac{1}{2}BC = r$.

Also solved by S. LEFSCHETZ, A. L. MCCARTY, GEORGE N. BAUER, HERMON L. SLOBIN, BARNEM LIBBY, RICHARD MORRIS, E. S. INGHAM, and HARVEY MANN.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

That workers in a given field of science or in closely related fields may render an important service by coöperation is one of the commonplaces in the practice of scientific discovery. This coöperation is realized in part through the general meetings in which those interested in the same or related subjects exchange views and present the results of their individual investigations. Those who are thrown together in any way further the same end through conversation; and useful exchange of thought often takes place by correspondence also.

It appears that there is an opportunity to enlarge this coöperation, and thus secure to science a greater advantage from it, by having some medium through which a certain type of questions can be presented to one's co-laborers and their reactions promptly secured. In the field of the mathematical sciences, one thinks immediately of the problems in *L'Intermédiaire des Mathématiciens*. We propose to establish in this journal also a department in which mathematical questions of this general kind are treated. From the account of our purposes, which we give below, it will be seen that we shall supply a need different from that which is provided in the *Intermédiaire*. In fact, so far as we know, there does not at present exist a medium of coöperation of the kind which we are about to establish.

Through this department of "Miscellaneous Questions" two purposes are to be served. One of these pertains more or less to pedagogical matters; the other relates entirely to scientific questions of a mathematical nature. As to our purposes in the former the reader is referred to the concluding paragraph under the heading "Problems and Questions" in the October number of the MONTHLY, from which the following is quoted:

> "Such questions might well refer, for example, to the selection of subject matter and method of presentation in courses in collegiate mathematics, and they might elicit replies in the form of a short symposium or they might even lead in some cases to extended articles. On the other hand, these questions might refer to difficulties of interpretation or to obscure steps in some demonstration or to anything whatsoever of mathematical import on which the questioner honestly seeks light."

As to the latter, the following remarks will explain our point of view: In many investigations both in mathematics and in physics there arise from time to time certain definite mathematical problems which are incidental to the main research in hand and must be solved before that research can be carried forward to completion. These problems often give special difficulty. Sometimes the investigator would be glad to have the assistance of others in their solution, either because they are difficult or because he prefers not to divert his attention from the main problem to solve the incidental one which arises in this way. One principal aim of this department is to secure and publish just such problems and their solutions. We shall also welcome similar ones which sometimes arise in connection with one's reading.

The mutual service which students and investigators in mathematics may render each other through such a medium is apparent to all who know the peculiar difficulties which such persons have to face. There is also another end of equal (and perhaps greater) importance which we hope may be served by this department. We desire to have proposed here well-defined mathematical problems which arise in physical investigations. To call wide attention to such problems will serve two useful purposes.

In the first place, the physicist may sometimes draw on the mathematician for assistance from his fund of knowledge about pure mathematics, and thus often secure readily the solution of his special incidental problem or an indication of its essential nature and difficulty. Thus, in the end, there may well be a considerable saving of time and energy.

In the second place, the mathematician cannot fail to profit largely by this interchange, as any one must see who remembers how often some important mathematical theory has been brought to notice and development through the persistent demand of physics for the means of solving her problems.

When a problem has been proposed to this department we shall undertake to supply the proposer with a solution as soon as possible after one has been received by us—sending it to him in advance of publication when that is feasible and is desired.

The undersigned will attempt to ascertain whether the problems which come to us are of sufficient interest or difficulty to justify examination by several persons. If some ready means of solving a particular problem is found, the proposer will be so informed as promptly as possible and a solution will usually be furnished him privately; and in such event the problem will be published only when it seems to be of interest in itself.

The first requisite for the usefulness of this department is to have problems and questions proposed; and, consequently, persons interested are urgently desired to send in their proposals at their early convenience. These should be accompanied by any remarks which may be useful to the editor in stating the question so that the reader may see why it is of interest to have an answer. Every communication should be signed by the author; his name, however, will be withheld if he so requests.

While the plans for this department were being considered two communications came to us, which are of the general nature wanted here; consequently, the questions involved are printed in this issue as 1 and 2. Answers are desired both to these questions and to others to be published later. It is obvious that a large part of the usefulness of this department will depend on the promptness with which answers to questions are supplied; and consequently our readers are requested to communicate replies as early as possible after a question is published.

All correspondence relating to this department should be addressed to R. D. Carmichael, 417 N. Lincoln Street, Bloomington, Indiana.

QUESTIONS.

1. In connection with the investigation of a problem in physics MR. LOUIS COHEN, Washington, D. C., requires to have expressed in the form of a Fourier's series,

(1) $$y = \Sigma A_n \cos nt + \Sigma B_n \sin nt,$$

the general solution of the differential equation

(2) $$\frac{dy}{dt} + (\alpha + \beta \cos t)y = \rho \cos t,$$

where α, β and ρ are constants. Is the solution of this problem contained directly in matter which has already been published? If not, how may one proceed to the determination of the constants A_n and B_n so that (1) shall be the general solution of (2)?

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

Note.—One very interesting and thought-arousing answer is given to this question in the paper by Professor Bussey in this issue. We shall welcome discussion along this line.—MANAGING EDITOR.

NOTES AND NEWS.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

Attention is called to questions one and two in this issue under the new department "Miscellaneous Questions," and to the introduction to this department by Professor R. D. Carmichael, who will have it in charge.

Dr. S. E. Rasor, of the Ohio State University, has been promoted to a full professorship in mathematics.

The Deutsche Mathematiker-Vereinigung held its annual meeting in Vienna on September 22–25, 1913. Altogether thirty-seven papers were presented.

Mr. George I. Gavett, University of Washington, has accepted an assistantship in the University of California for the present year.

At Lehigh University, Joseph B. Reynolds has been made assistant professor of mathematics and astronomy.

Mr. C. W. Wester, graduate student at the University of Chicago, has been appointed to an instructorship in mathematics at the University of Iowa.

At the University of Colorado the following have been appointed instructors in mathematics for the coming year: Florence Kendall, B.A., University of Colorado, 1912; W. E. Edington, B.A., Indiana State Normal, and last year a scholarship student at the University of Chicago, majoring in mathematics.

Dr. H. L. Rietz, associate professor of mathematics in the University of Illinois, has been promoted to a full professorship in the same institution. Mr. H. C. Zeis has been appointed assistant in mathematics in place of Mr. G. E. Carscallen, who accepted a professorship in Hiram College.

The Proceedings of the Fifth International Congress of Mathematicians held at Cambridge in August, 1912, have been edited by the general secretaries of the Congress, E. W. Hobson, professor of pure mathematics in the University of Cambridge, and A. E. H. Love, professor of natural philosophy in the University of Oxford.

The Proceedings appear in two volumes; price 30 shillings.

On the 25th of last May, the "friends and admirers, collaborators and students" of Professor Felix Klein presented to him his protrait, painted by Max Liebermann. The published presentation address is followed by a large number of names of mathematicians from various countries. The names of the following Americans appear in the list: M. Bôcher, F. N. Cole, P. Field, J. C. Fields, F. Franklin, A. B. Frizell, M. W. Haskell, G. A. Miller, E. H. Moore, W. F. Osgood, D. E. Smith, V. Snyder, J. H. Tanner, H. W. Tyler, E. B. VanVleck and F. S. Woods. In reply to this address Professor Klein stated that the real motive of his life had always been the collaboration with those engaged in similar work.

Professor J. F. Downey, University of Minnesota, expects to retire at the end of the present academic year. He is professor of mathematics and dean of the college of science, literature, and arts.

The Spanish Mathematical Society, which was organized at Madrid a little more than three years ago, has now 470 members, according to the July, 1913, number of *Revista de la Sociedad Matematica Española*. The membership fee for foreign members is 18 pesetas (about $3.60) per year. Members receive without extra charge a copy of the "Revista." This elementary journal appears monthly in the Spanish language, and is published for the society by an editorial committee and the secretaries. The headquarters of the society are at San Bernardo, 51, Madrid, Spain.

The 26th fascicule of the *Encyclopédie des Sciences Mathématiques* was published on June 30, 1913. It contains 160 pages, and is devoted to partial differential equations. The first 55 pages, edited by G. Floquet of Nancy, are concerned with the study of general properties and linear equations of the first order. The remainder is edited by E. Goursat of Paris, and is devoted to non-linear equations of the first order, and to equations of a higher order.

We learn from the widow of Dr. Alexander Macfarlane, of Chatham, Ontario, that he died on August 28, 1913, after an illness of six months from valvular heart disease. He was at one time instructor in physics in the University of Edinburgh and later held the professorship of physics in the University of Texas. He had been a subscriber to the MONTHLY from the beginning, and one of its warmest supporters. Our sympathies are extended to Mrs. Macfarlane in her bereavement.

At Rutgers College, Professor Richard Morris has been given the title "professor of mathematics" in place of "professor of mathematics and graphics." Also two new appointments have been made to the rank of assistant professor, namely, Dr. W. B. Stone and Dr. S. E. Brasefield.

Announcement has been made by the department of mathematics, Adelbert College, Western Reserve University, of a course in the history and teaching of elementary mathematics for 1913–14. The course is intended especially for instructors in elementary mathematics and persons preparing to become teachers, and is open to both men and women.

The death is announced of Mr. T. N. HAAN, of Mohawk, Tenn. He was an enthusiastic supporter of the MONTHLY and was interested in mathematics especially on the practical side. He left an extensive library and a manuscript on mathematical lines which he had prepared for publication.

The mathematics section of the California High School Teacher's Association held two meetings during the Conference in Berkeley last July. At the first meeting a committee was appointed to devise means of raising the standard of scholarship throughout the state. This committee recommended that, in the larger cities, circles be formed for the study and solution of problems. Also that every teacher become familiar with at least one first class journal, such as the AMERICAN MATHEMATICAL MONTHLY, and if possible take up the study of *Mathematical Monographs*, edited by J. W. A. Young. As a result of this recommendation, the study of two or more of these monographs is included among the subjects in which examinations will be given in the university extension department of the University of California. The second meeting emphasized the need for arithmetic which concerns the investments and expenditures of daily life. It constituted a plea for more such courses in the high school to meet the future needs of the average boy and girl. Dr. HENRY W. STAGER, of the Junior College at Fresno, was elected president for the ensuing year.

As the MONTHLY is nearing the end of the first year in its new form, attention is called to the classification of articles so far published to be found on the supplementary pages of the last issue. Readers will be pleased to know that the subscription list has been nearly doubled during the year. However, this is only a beginning and the editors bespeak the coöperation of all friends in still further extending the list. An important field where such influence may be effective is in connection with school and college libraries and public libraries. Sample copies of this issue have been sent to a large number of such libraries not now on the mailing list.

It will greatly facilitate the business management if all renewals for 1914 may be received by the Treasurer before January. Please note that the date "Jan. 14," on the address label signifies the expiration of your subscription with the December issue.

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NUMBER SYSTEMS OF THE NORTH AMERICAN INDIANS.

BY W. C. EELS, Whitworth College.

II. SYSTEMS OF NUMERATION.

Counting cannot be carried far by the use of successive unrelated terms or symbols for each number. For higher terms some system of compounding is necessary, the usual method being by reference to some stopping point or *base* which is thought of as a new starting point. The choice of this base is of fundamental importance in the development of a true *number system* as distinguished from a mere *series of numbers*. The choice of this base seems usually to be related to primitive finger counting. One hand, two hands, or all the fingers and toes, are the three most natural stopping points, leading respectively to quinary, decimal, and vigesimal systems as illustrated below. There is much variation in the choice of these bases and in the way in which they are built up into systems for actual use after the base has been established. Thus a pure quinary system is rare, usually merging into a decimal or vigesimal one for the higher numbers. Accordingly in the list given below of the number of instances of each system found, quinary and quinary-decimal systems have been classed together, and similarly vigesimal and quinary-vigesimal.[1] The numbers found are: decimal 146, quinary and quinary-decimal 106, vigesimal and quinary-vigesimal 35, quaternary 15, ternary 3, octonary 1 (binary 81). The binary instances given in parantheses refer simply to languages in which the duplicative principle (already discussed) occurs. These may be thought of as traces of a binary system. But there are no binary systems in the true sense of the word.

1. Decimal Systems. Many of the American Indian systems have decimal scales as regular and as complete as our own, extending to quite high limits.

[1] Probably several systems have been wrongly classified, due to inadequate data. Many authorities give the numerals from 1 to 10 and then for 100, or perhaps by tens to 100. The greatest error is probably in the classification of too many languages as decimal. This has been done in some cases from comparison with cognate languages, where the direct data were insufficient. Some have been called quinary-decimal, where further data would probably show they were actually quinary-vigesimal. These variations in detail will probably not seriously affect the general conclusions stated.